Angular Momentum and the Rolling Chain

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Contents

1 Introduction

The demonstration proposed in this report is one in which a closed loop of chain is fitted onto a wooden cylinder and spun up to a large rotational speed with an electric motor. The chain is then nudged off the cylinder with a stick or screwdriver and will retain its circular shape and roll across the floor in a straight path until eventually coming to rest in a pile.

There are two complimentary explanations for the behavior of the chain: centripetal force and conservation of angular momentum. The chain retains its shape because of the inertia of each of its links which tend to move in a straight line tangent to the circle. This tendency is ascribed to a centripetal force acting toward the center of the circle in order to keep the chain from flying apart. The conservation of angular momentum requires that the chain continues rotating (conserving both the magnitude and direction of total angular momentum) until sufficient frictional torque brings it to rest.

This report will begin by deriving the equations for circular and angular motion relevant to the demonstration. The report then uses these principles and applies them to the rolling chain in order to calculate theoretical quantities of interest. The report will conclude with a brief outline of a presentation and a feasibility plan for project completion by the recommended deadline.

2 Theory

The basic equations for circular and angular motion are derived from first principles starting with uniform circular motion and ending with the conservation of angular momentum. Many of the derivations are reduced to their simplest form in an attempt to offer clear and concise explanations without the complexity of calculus.

2.1 Uniform Circular Motion

Any object that moves in a circular path with speed v has a velocity vector whose direction is continually changing but magnitude remains constant. Since the direction of the velocity vector is constantly changing in order to remain tangential to the circular path, the object must experience an acceleration as acceleration is defined as the rate of change of velocity. Hence, any object undergoing uniform circular motion experiences an acceleration even though its speed remains constant.

If we look at the change in velocity by comparing two infinitesimally close velocity vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, we can see that because both vectors must be tangential to the circle, then $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ must point toward the center of the circular path. Because the acceleration $\mathbf{a}$ must also point in the same direction as $\Delta\mathbf{v}$, the acceleration experienced is called a **centripetal acceleration**.

We can derive the magnitude of the centripetal acceleration by noting that the time it takes a point to complete a circular path of radius r (also known as the object's **period**, T) can be written as $T = \frac{2\pi r}{v}$. Similarly, this quantity can also be written as $T = \frac{2\pi v}{a}$. Equating these two expressions for T allows us to solve for a, the magnitude of the centripetal acceleration:

$$a = \frac{v^2}{r}.$$

Thus, an object moving in a circular path with radius r at a constant speed v experiences an acceleration whose direction is toward the center of the circular path and magnitude is $\frac{v^2}{r}$. By Newton's Second Law, an object of mass m traveling in a circular path will experience a **centripetal force**, $\mathbf{F}$, toward the center of the path whose magnitude is given by

$$F = ma = m\frac{v^2}{r}.$$

2.2 Angular Motion

It is useful to define the measurement of angles in **radians** instead of degrees when working with circular motion. One radian is defined as an angle subtended by an arc with length equivalent to the circle's radius. Any angle θ can be defined in radians as

$$\theta = \frac{l}{r}$$

where l is the length of the arc along the circumference of the circular path and r is the path's radius. Note that the radian is actually a dimensionless quantity as it simply the ratio of two lengths. Also note that degrees can be converted to radians (or vice versa) using the relationship

$$360^o = 2\pi \text{ radians}$$

where the left-hand side is the number of complete degrees in a circle and the right-hand side is the number of complete radians. From this, we can see that 1 rad $\approx 57.3^o$.

Every point on a rigid body rotating about a fixed axis moves in the same circular path: each point will sweep out the same angle θ in the same time. When a rigid body body rotates in a circular path from some initial angle θ_1 to some final angle θ_2 in some time Δt, we can define a vector quantity called the **angular velocity**, ω, whose magnitude is given as

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

in the limit as $\Delta t \to 0$ (where $\Delta\theta = \theta_2 - \theta_1$). Each point on the body moves with the same angular velocity. Note that angular velocity is expressed in units of radians per second. To find the direction of the vector quantity ω we use the **right-hand rule**. When the fingers of the right hand are curled around the axis of rotation and point in the direction of rotation, then the thumb gives the direction of ω. Notice that no point on the rotating body actually moves in the direction of ω as it is always parallel to the axis of rotation.

If we consider a point at a distance r from the axis of rotation that is rotating with angular velocity magnitude ω, we can say that the point will have a linear velocity $\mathbf{v}$ tangential to the circle with magnitude

$$v = \omega r.$$

An object's period of rotation T is inversely related to its **frequency** f as in $T = \frac{1}{f}$ where f is often expressed in revolutions per second. We can now relate an object's angular velocity magnitude to its frequency as

$$\omega = 2\pi f$$

since one revolution corresponds to an angle of 2π radians.

2.3 Angular Momentum

The linear momentum quantity $\mathbf{p} = m\mathbf{v}$ has a rotational analog known as **angular momentum**. This vector quantity, often expressed as $\mathbf{L}$, takes the form

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

for an object moving in uniform circular motion at a radius r. The units of angular momentum are kilograms meters squared per second. To find the direction of the angular momentum vector, we also use the right-hand rule. When the fingers point in the direction of $\mathbf{r}$ and they curl toward the direction of $\mathbf{v}$, the right thumb will point in the direction of $\mathbf{L}$. For a rigid body rotating about a fixed axis, the direction of the angular momentum vector can be taken to be the same as the direction of the angular velocity vector, ω. This is true only if the axis of rotation is perpendicular to the plane of the body (like a bicycle wheel).

There also exists an angular analog to the conservation of linear momentum. Like linear momentum, angular momentum is always conserved if the net external torque acting on a system is zero. We can rewrite the angular momentum of a system in a more suggestive manner in order to make its conservation law look similar to the one for linear momentum ($\mathbf{p} = m\mathbf{v} = \textit{constant}$ where both the magnitude and direction of $\mathbf{p}$ are conserved). The first thing we need to do is substitute $\omega = \frac{v}{r}$ into our equation for angular momentum. In order to neglect the cross-product, we can also take the direction of $\mathbf{L}$ to be the same as the direction of ω for the reasons described above. So our equation for $\mathbf{L}$ simplifies to:

$$\mathbf{L} = mr^2\omega.$$

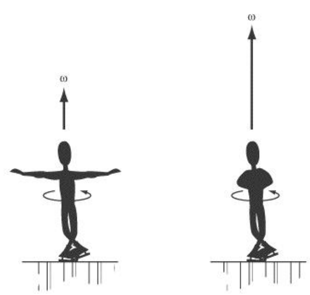
Now, if we define a quantity called the **moment of inertia**, I, and set it equal to mr^2 in the formula above for $\mathbf{L}$, we can write

$$\mathbf{L} = I\omega$$

which looks very similar to the equation for linear momentum. Hence, the conservation of angular momentum can be written in the form

$$I\omega = \text{constant}.$$

And this is true for all rigid bodies rotating about a fixed axis when there is no external torque. The fact that we set $I = mr^2$ in this derivation is specific to a hoop of radius r rotating about a fixed axis. The moments of inertia for other rotating bodies are different although the general formula for angular momentum of $\mathbf{L} = I\omega$ remains the same (the general formula for the moment of inertia is $I = \sum m_i r_i^2$). Since our demonstration involves a rolling chain, we can approximate it's moment of inertia with the moment of inertia for a hoop rotating about an axis drawn through its center. We can see the conservation of angular momentum in action if we picture a figure skater doing on spin on ice: the skater rotates faster when more mass is concentrated toward the axis of rotation.



3 Demonstration

The demonstration of the rolling chain is discussed with reference to many of the derivations made above. Quantities of interest, such as the translational velocity of the rolling chain, are calculated. The section concludes by attributing the behavior of the rolling chain to the two complementary principles at work: centripetal force and conservation of angular momentum.

3.1 Apparatus

In this demonstration, a flexible, circular loop of chain is spun up to a large angular velocity and fitted onto a wooden cylinder by using a high speed AC motor. When the chain is spinning fairly fast, it is then nudged off the cylinder and proceeds to roll along the floor and over obstacles, maintaining its circular shape, until sufficient frictional torque brings it to rest.

The materials required for the apparatus are as follows: a loop of bicycle chain with inner diameter 20 cm, a short wooden cylinder of diameter 20 cm, an AC electric drill, a screwdriver, and a speed sensor. The wooden cylinder is fitted into the end of the electric drill which will be used to spin the chain up to approximately 1000 revolutions per minute. Once at this angular velocity, the chain will be worked off the cylinder by using the screwdriver. When it is released, the chain will grab the floor and roll along undergoing pure rotation without slipping like a rigid wheel. The speed of the rolling chain will be measured in order to compare both the theoretical and experimental speed of the chain in a class discussion. A picture of the apparatus is shown below.

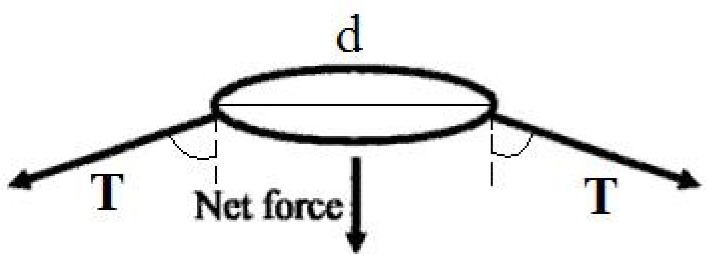
3.2 Application

This demonstration illustrates many of the physical principles of circular motion discussed earlier. For a circular object spinning at 1000 revolutions per minute, we can calculate the speed of the chain after it leaves the wooden cylinder using our equation relating angular velocity and linear velocity. We know that $f = \frac{1000}{60}$ revolutions per second and $\omega = 2\pi f = 2\pi\frac{1000}{60}$ radians per second. Thus, we can calculate the translational speed after the chain leaves the cylinder as

$$v = \omega r = (2\pi\frac{1000}{60} \text{ rads/s})(0.1 \text{ m}) = 10.5 \text{ m/s}$$

where the units of v are in meters per second. This speed is roughly the same as the average speed of an Olympic sprinter running the 100 m dash.

Each link on the bicycle chain acts like a point that follows a circular path. In order for the chain to retain is circular shape while it is rolling along the ground, each link must experience a net force toward to center of the chain resulting from the two tension forces acting on it by neighboring links. This net force toward the center of the circular path is equivalent to the centripetal force. Below is an image of the free-body diagram of a single chain link.



To calculate the value of the tension force, **T**, that a chain link experiences by each neighboring link, we need some simple values. A chain link, measured from pin to pin, is 1.27 cm long (this corresponds to the value for d in the image above). If we are going to use a 20 cm diameter (which is approximately a 60 cm circumference) chain, then we need 50 chain links which will form an almost circular 50-sided polygon. Using the interior angles of a 50-sided polygon, the angles drawn between the tension forces and the net centripetal force are 82.8^o in the image above. The mass of each chain link is 2.8 g.

We know that each link travels in a circular path with linear velocity vector, $\mathbf{v}$, tangential to the path. The speed v of each link is 10.5 m/s. We note that there are a total of two different forces acting on each chain link: gravitational force and centripetal force. The gravitational force can be calculated as $F_g = mg = 0.0274$ N. The centripetal force (due to the two tension forces from neighboring links) is calculated as $F_c = \frac{mv^2}{r} = 3.09$ N. Since $F_c >> F_g$ for each link, we can neglect F_g when determining the magnitude of the tension force from each neighboring link.

Since the net force due to the two tension forces must point toward the circle's center, we can see that the tangential components of the tension forces cancel. The net force we are interested in is then simply the sum of the radial components of the two tension forces.

This net force must equal the centripetal force calculated above. Thus, we can write

$$2T\cos 82.8^o = \frac{mv^2}{r} = 3.09 \text{ N}$$

and solving for T, we find $T = 12.3$ N. Because of the small angle between each link, the tension force due to each neighboring link is nearly 4 times the total centripetal force.

The behavior of the chain can also be explained in terms of the conservation of angular momentum. Once the chain is spinning at a high revolutions per minute, its angular momentum will be conserved even after it has left the cylinder and begins to roll along the floor. The total angular momentum of the chain immediately before and just after it leaves the cylinder must be equal. The value for the chain's angular momentum can be calculated as

$$\mathbf{L} = mr^2\omega = (0.14 \text{ kg})(0.1^2 \text{ m}^2)(2\pi\frac{1000}{60} \text{ rads/s}) = 0.15 \text{ kg m}^2\text{/s}$$

where the mass used here is the total mass of all 50 links in the chain. This quantity for angular momentum is conserved once the chain is released from the wooden cylinder, which offers a complementary explanation as to why the chain continues to roll until sufficient frictional torque brings it to rest.

Due to the law of conservation of angular momentum, not only is to the magnitude of the angular momentum conserved when the chain leaves the wooden cylinder, but its direction is conserved as well (since angular momentum is a vector quantity). In this demonstration, the reason as to why the chain continues to roll in a line parallel to its initial position on the wooden cylinder is because the direction of the chain's angular momentum is conserved. This can be thought of more intuitively if we picture the chain flying off the wooden cylinder horizontally; the chain will still retain its circular shape and will fly across the room like a saucer instead of rolling along the ground.

4 Conclusion

This demonstration will be presented to students in high school classrooms ranging from Physics 11 to Physics 12. In order to convey the importance of the physical principles behind circular and angular motion, many examples will be given in either the form of questions or simple, supplementary demonstrations. Students will be asked to participate in these demonstrations either physically or verbally.

During the rolling chain demonstration, the speed of the chain will be measured in order to compare an experimental speed with the theoretical one derived in class. A difference between the two should conclude the presentation with a discussion on other factors at work that we have neglected, such as friction and the slight flattening of the chain as it rolls along the floor.

References

[1] Douglas Giancoli, *Physics: For Scientists and Engineers*. Prentice Hall, New Jersey, 3rd Edition, 2000.