ANGULAR MOMENTUM AND THE ROLLING CHAIN

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UBC Physics and Astronomy
Physics 420
2008-2009

CONTENTS

LINEAR MOMENTUM

Q: "Why did the chicken cross the road?"

As Isaac Newton would have said:

A: "Chickens at rest tend to stay at rest, chickens in motion tend to cross roads."

LINEAR MOMENTUM

$$\mathbf{p} = m\mathbf{v}$$

The **momentum** of a body is defined as the **product** of its **mass** and its **velocity**. Momentum is a **vector** quantity. The **direction** of the momentum vector is the same as the direction of the velocity vector.

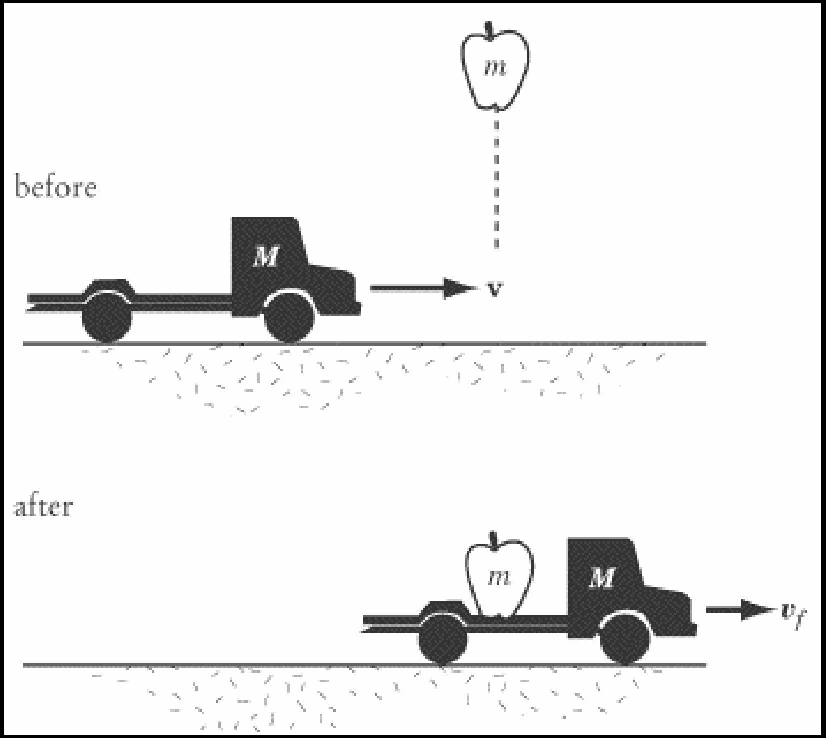
- units: [N s] or [kg m/s]
- vector: direction and magnitude
- conserved: the total momentum of any closed system does not change if there are no external forces acting on it

$$\mathbf{p}_{\text{before}} = \mathbf{p}_{\text{after}}$$

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

COLLISION PROBLEM

What is the final velocity ($\mathbf{v}_f$) of the truck and apple after the collision in terms of the initial velocity ($\mathbf{v}$), the mass of the apple (m), and the mass of the truck (M)?



SOLUTION

Using the conservation of linear momentum:

$$\mathbf{p}_{\text{before}} = \mathbf{p}_{\text{after}}$$

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

Plugging in the variables we know:

$$M\mathbf{v} + m(0) = (M + m)\mathbf{v}_\text{f}$$

Solving for $\mathbf{v}_\text{f}$:

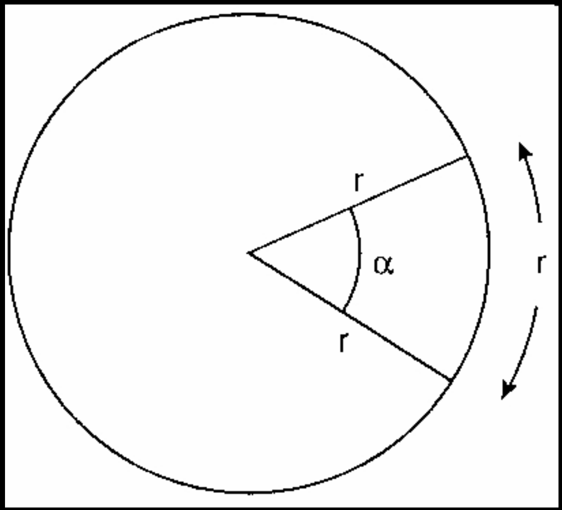
$$\mathbf{v}_\text{f} = M\mathbf{v} \,/\, (M + m)$$

Let's also look an example of two carts.

CIRCULAR MOTION

Some basic relationships:

- There are 2π radians in a circle (2π = 360˚)
- One **radian** is defined as an angle subtended by an arc with length equal to the circle's radius



CIRCULAR MOTION

- Objects moving in a straight line have a velocity vector (**v**) that points in the direction of travel and has units of m/s

- The magnitude of the velocity vector is defined as the change in distance traveled over the change in time:

$$v = \Delta d/\Delta t$$

CIRCULAR MOTION

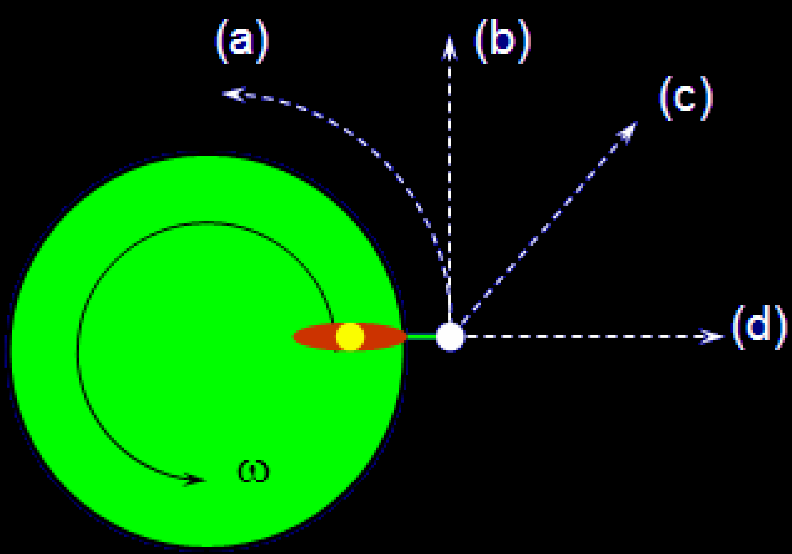
- Things moving in a circle have an angular velocity (ω) which has units of rad/s
- The magnitude of the angular velocity is defined as the change in angle over the change in time:

$$\omega = \Delta\theta/\Delta t$$

- The direction of the angular velocity vector is parallel to the axis of rotation

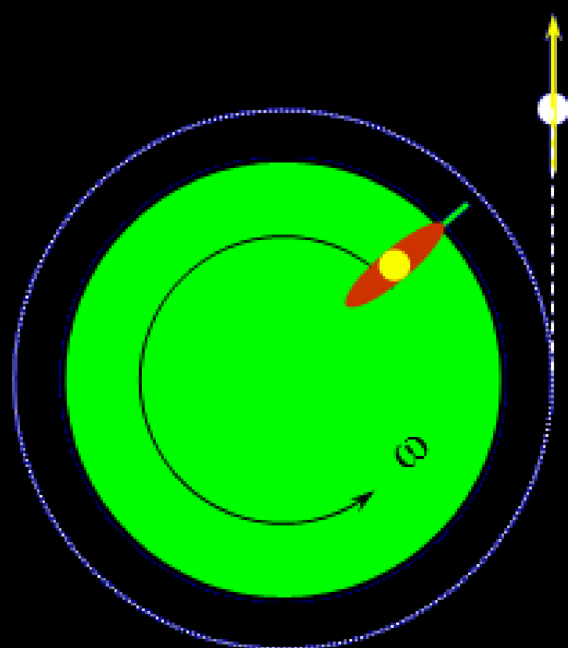
ROTATION PROBLEM

A girl is riding on the outside edge of a merry-go-round turning with constant ω. She holds a ball at rest in her hand and releases it. Viewed from above, which of the paths shown below will the ball follow after she lets it go?



SOLUTION

Just before release, the velocity of the ball is tangent to the circle it is moving in. After the release, it keeps going in the same direction since there are no forces acting on it to change this direction. Whose law is this?



NEWTON'S FIRST LAW

"Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it."

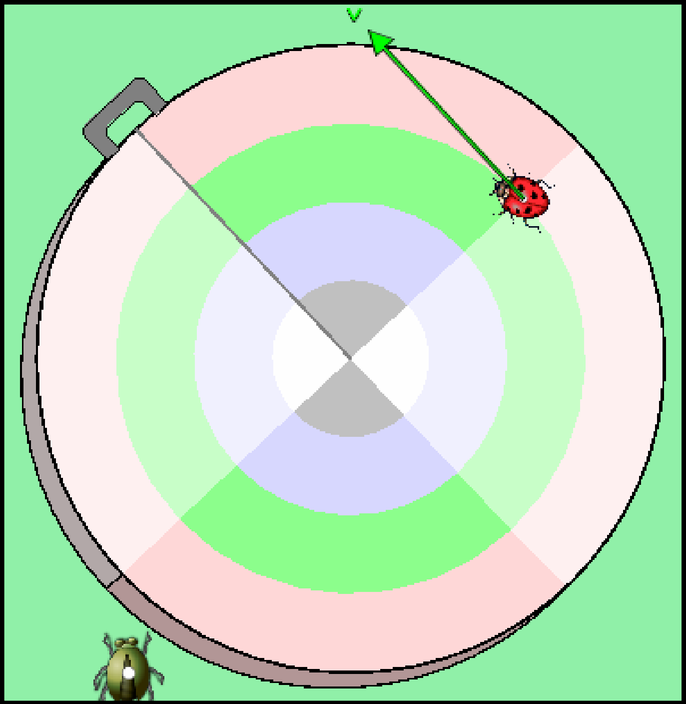
CIRCULAR MOTION

- If we consider a point at a distance *r* from the axis of rotation that is rotating with angular velocity ω, we can define the magnitude of the tangential velocity as:

$$v = \omega r$$

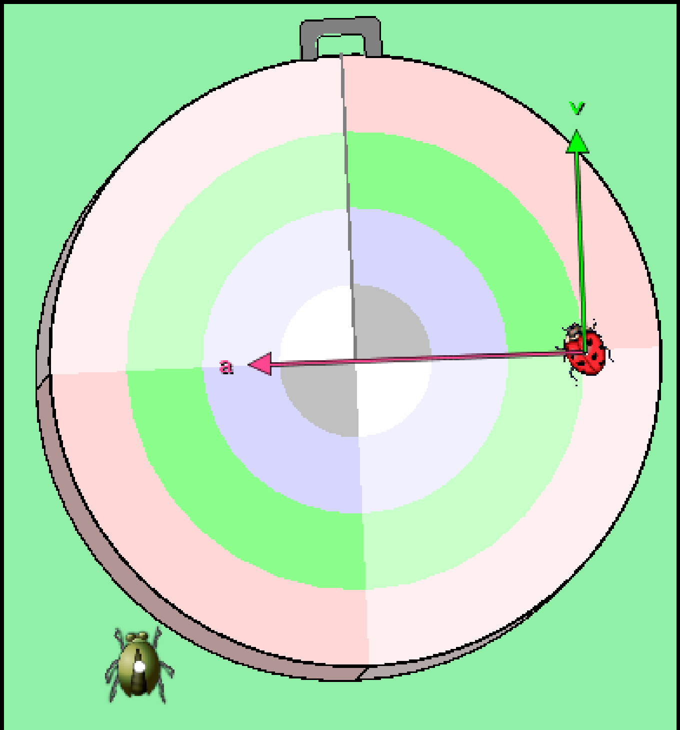
CENTRIPETAL ACCELERATION

A bug is spinning on a platform with constant speed (tangential velocity is the green vector). What is the direction of the bug's acceleration?



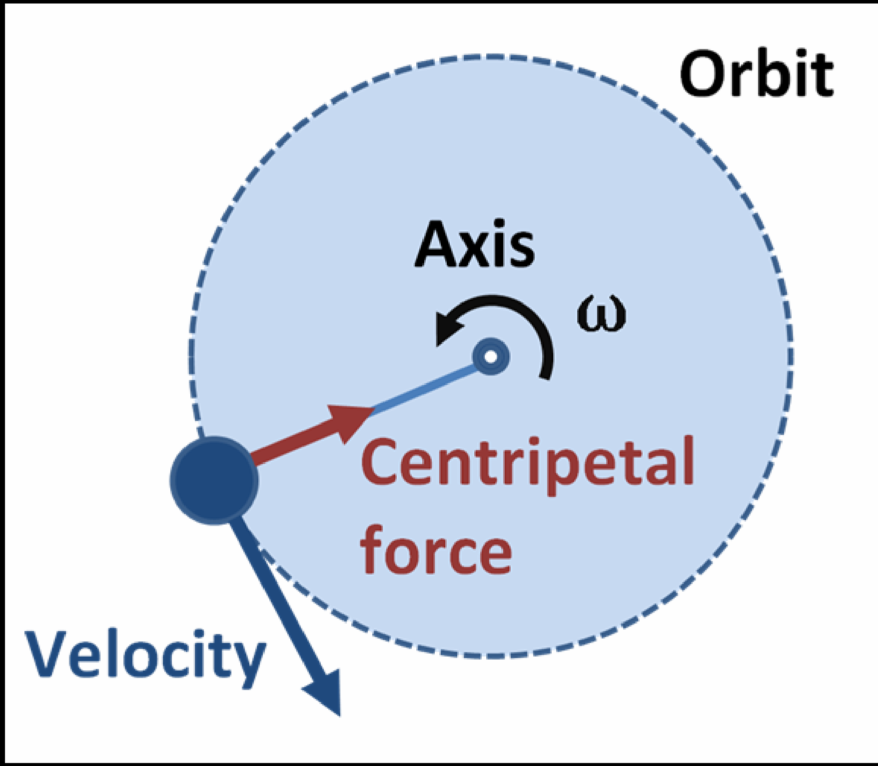
CENTRIPETAL ACCELERATION

The bug's acceleration vector always points radially (towards the center of the circular path) for a constant speed.



CENTRIPETAL ACCELERATION

When a body moves in a circular path, it experiences an acceleration that points towards the center of the circular path.

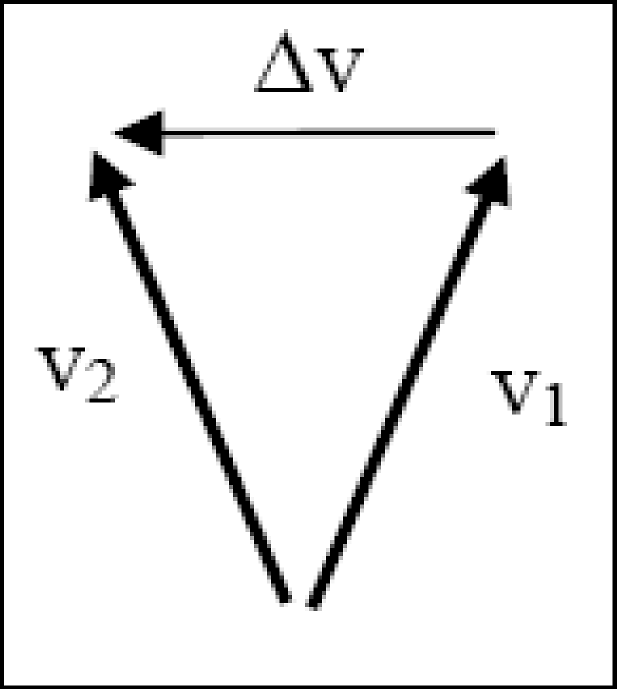


Why is this?

(Hint: What is the definition of acceleration?)

CENTRIPETAL ACCELERATION

A body experiences an acceleration towards the center of it's circular path because it's velocity vector is constantly changing.



APPLICATION

Let's play with a demonstration which illustrates the principles of **angular motion** such as **angular velocity** and **angular acceleration**.

Click here

CENTRIPETAL ACCELERATION

The formula for centripetal acceleration is:

$$\mathbf{a} = \mathbf{v}^2/\mathbf{r}$$

We can use Newton's Second Law to define the centripetal force as:

$$\mathbf{F} = m\mathbf{a} = m\mathbf{v}^2/\mathbf{r}$$

ANGULAR MOMENTUM

$$\mathbf{L} = \mathbf{rp} = \mathbf{r}m\mathbf{v}$$

- units: [N m s] or [kg m^2/s]
- vector: direction and magnitude
- conserved: a system's angular momentum stays constant unless an external force acts on it

$$\mathbf{L}_{before} = \mathbf{L}_{after}$$

$$(\mathbf{r}m\mathbf{v})_{before} = (\mathbf{r}m\mathbf{v})_{after}$$

ANGULAR FREQUENCY AND MOMENT OF INERTIA

Redefine angular frequency ω as:

$$\omega = \mathbf{v}/\mathbf{r}$$

Then we can write **L** as:

$$\mathbf{L} = m\mathbf{r}^2\omega$$

If we define a quantity $\mathbf{I}$ (moment of inertia) as

$$\mathbf{I} = m\mathbf{r}^2$$

Then we can write the angular momentum as:

$$\mathbf{L} = \mathbf{I}\omega$$

And this is the angular analogue to $\mathbf{p} = m\mathbf{v}$!

ANGULAR MOMENTUM VECTOR

magnitude: **L** = **r***m***v** = $I\omega$

direction: the direction of **L** is found using the right-hand-rule on the product of **r** and **v**.

Take your right hand, point your fingers in the direction of **r** and curl them towards to the direction of **v**. Your right thumb points in the direction of **L**.

CONSERVATION OF ANGULAR MOMENTUM TOOLKIT

Now that we have derived an angular equivalent of linear momentum we can conserve this quantity in many types of problems.

Remember, as in linear momentum, both the magnitude and the direction of angular momentum are always conserved.

$$\mathbf{L}_{before} = \mathbf{L}_{after}$$

$$(\mathbf{r}m\mathbf{v})_{before} = (\mathbf{r}m\mathbf{v})_{after}$$

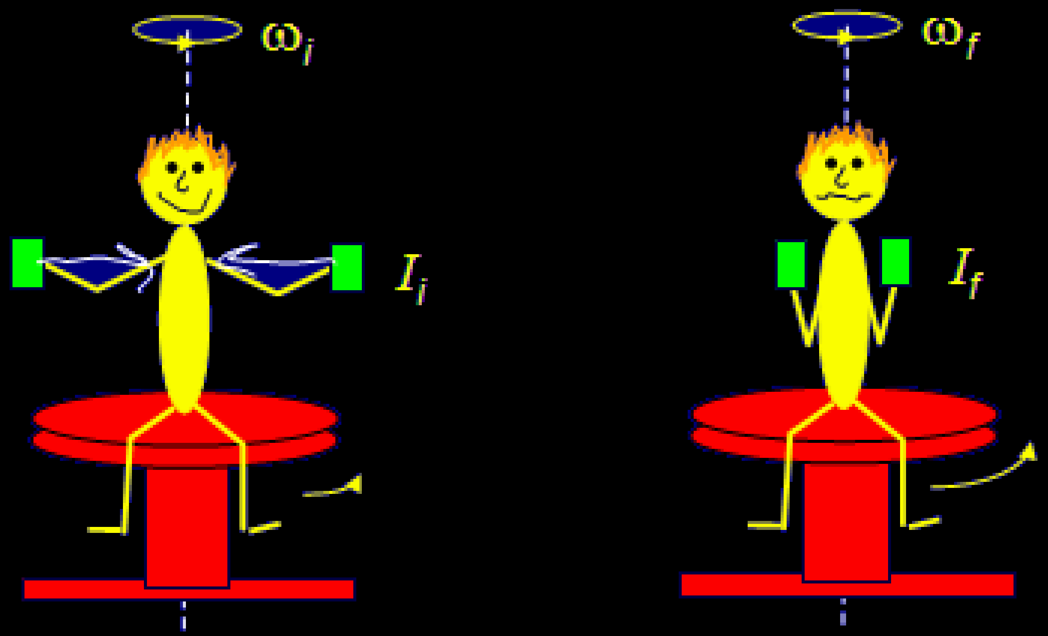
$$(\mathrm{I}\omega)_{before} = (\mathrm{I}\omega)_{after}$$

where $\mathrm{I} = m\mathbf{r}^2$ and $\omega = \mathbf{v}/\mathbf{r}$

CONSERVING ANGULAR MOMENTUM MAGNITUDE

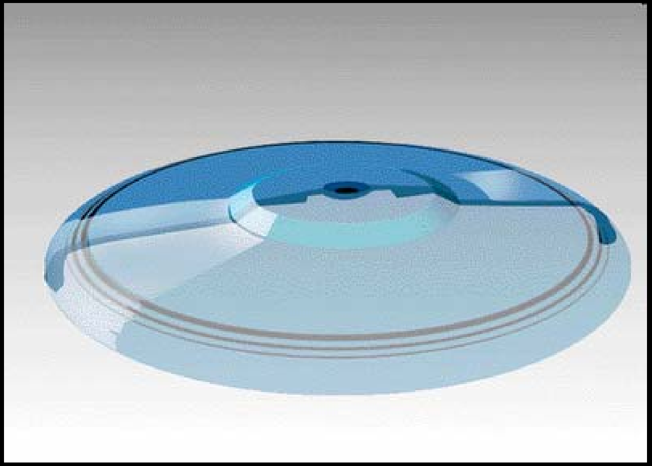
Do you ever notice that you seem to spin faster on a chair when your arms are closer to your body? It's true!

This is because you are decreasing your moment of inertia – and so your angular velocity must increase in order to conserve your total angular momentum! (video)



CONSERVING ANGULAR MOMENTUM DIRECTION

Why is it that when you a throw a Frisbee, it tends to stay flat?



This is because the direction of the angular momentum vector is always conserved!

DEMONSTRATION

DISCUSSION

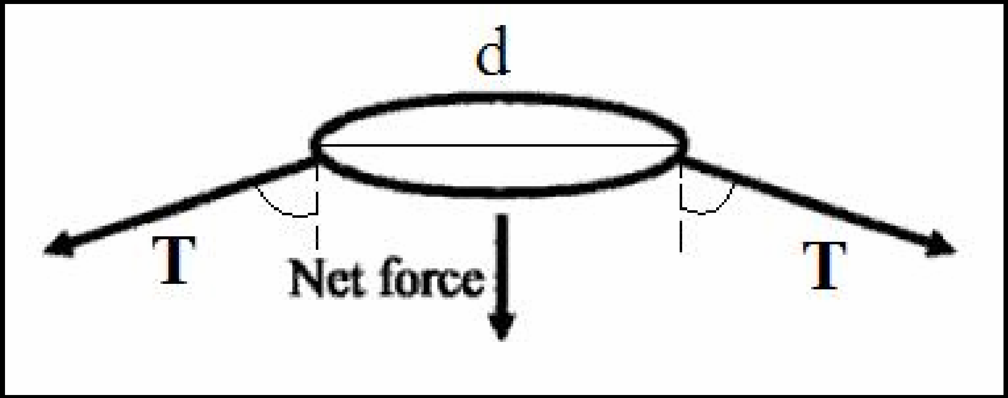
- What happened?
  - The chain retained its circular shape and rolled along the ground!

- Why did this happen?
  - Each link experiences a net centripetal force, and total angular momentum is conserved when the chain is released!

EXPLANATION

- Each chain link acts like a point that follows a circular path when spinning on the wheel
- The inertia of each of its links causes them to move in a straight line tangent to the circle, even when released from the wheel
- For the entire chain to retain its circular shape, each link experiences a net force toward the center due to tension forces from neighbouring links

FREE-BODY DIAGRAM



The net tension force toward the center of the chain is equivalent to the centripetal force.

This force keeps the chain from flying apart when released!

FURTHER EXPLANATION

Both the magnitude and direction of angular momentum are conserved after the chain is released:

- The total angular momentum immediately before and immediately after it is released must be equal
- The direction of angular momentum is also conserved as the chain rolls parallel to its initial position before being released

CALCULATIONS

A 20-cm-diameter wooden cylinder with circular chain rotating at 1000 rpm will give a final translational velocity of:

$\mathbf{v} = \omega \mathbf{r}$ = 2π * 1000 rev./min * (1 min/60 sec) * (0.1m)

= 10.5 m/s

The chain will continue to roll across the ground, until friction finally brings it to rest.

CONCLUSION

- Centripetal acceleration is directed towards the center of the circular path because of an object's constantly changing velocity vector.
- Angular momentum is a vector quantity – it has a magnitude and direction:
  - It's magnitude is **L** = **r***m***v** = $\mathrm{I}\omega$
  - And it's direction can be found using the right-hand-rule
- Both magnitude and direction are conserved.

QUESTIONS?