Projectile Motion: The Pneumatic Canon

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Equations of Motion

$v_f = v_0 + at$

$v_f^2 = v_0^2 + 2ad$

$d = (½)at^2 + v_0t$

$d = (½)t(v_f + v_0)$

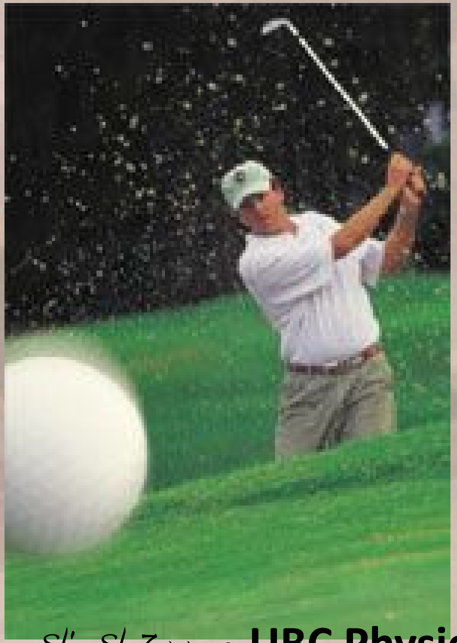
How are they derived → **calculus**

Application of Kinematics

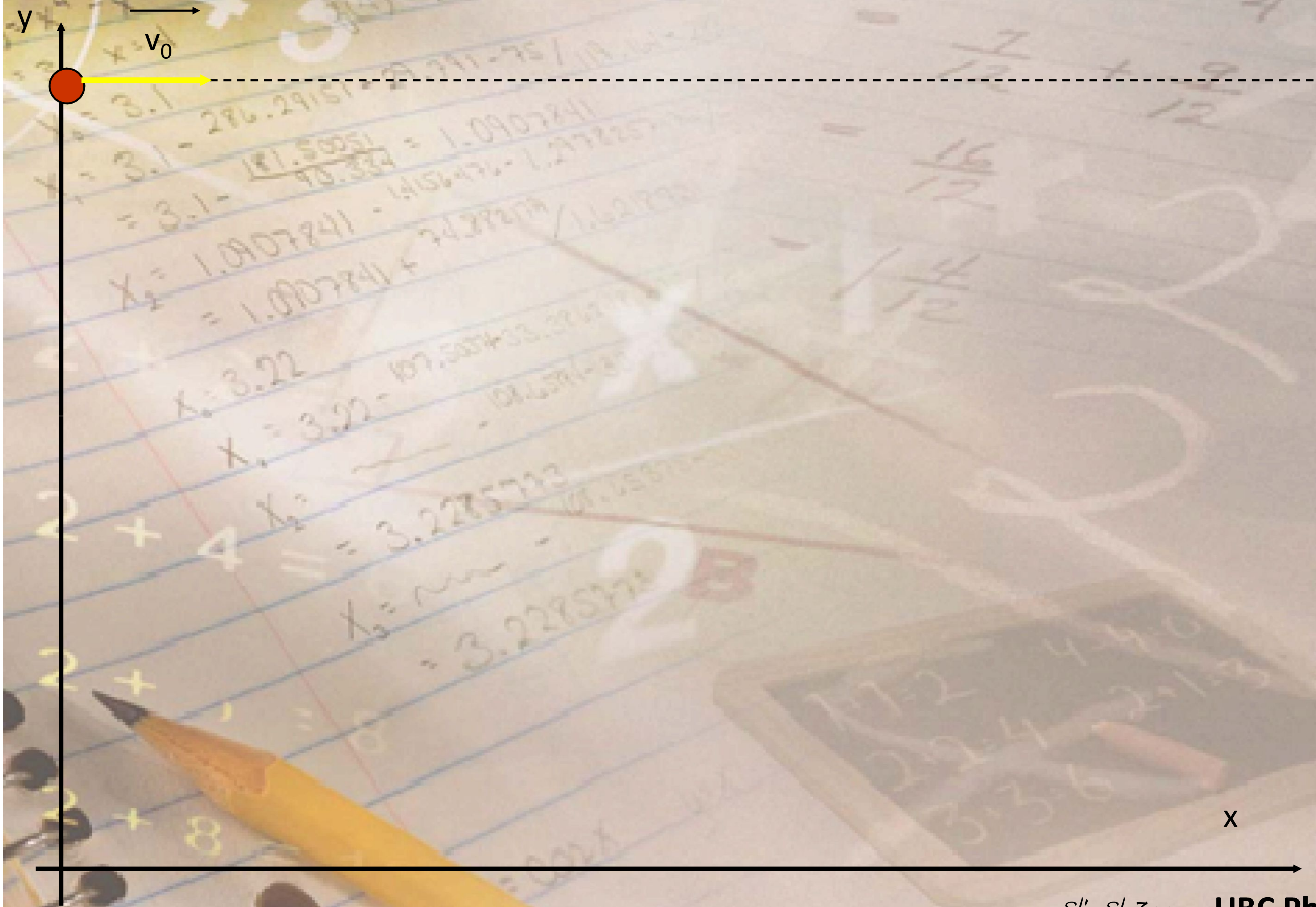
- Projectile Motion:

Motion through the air without propulsion

- Examples:



Motion of Objects Projected Horizontally



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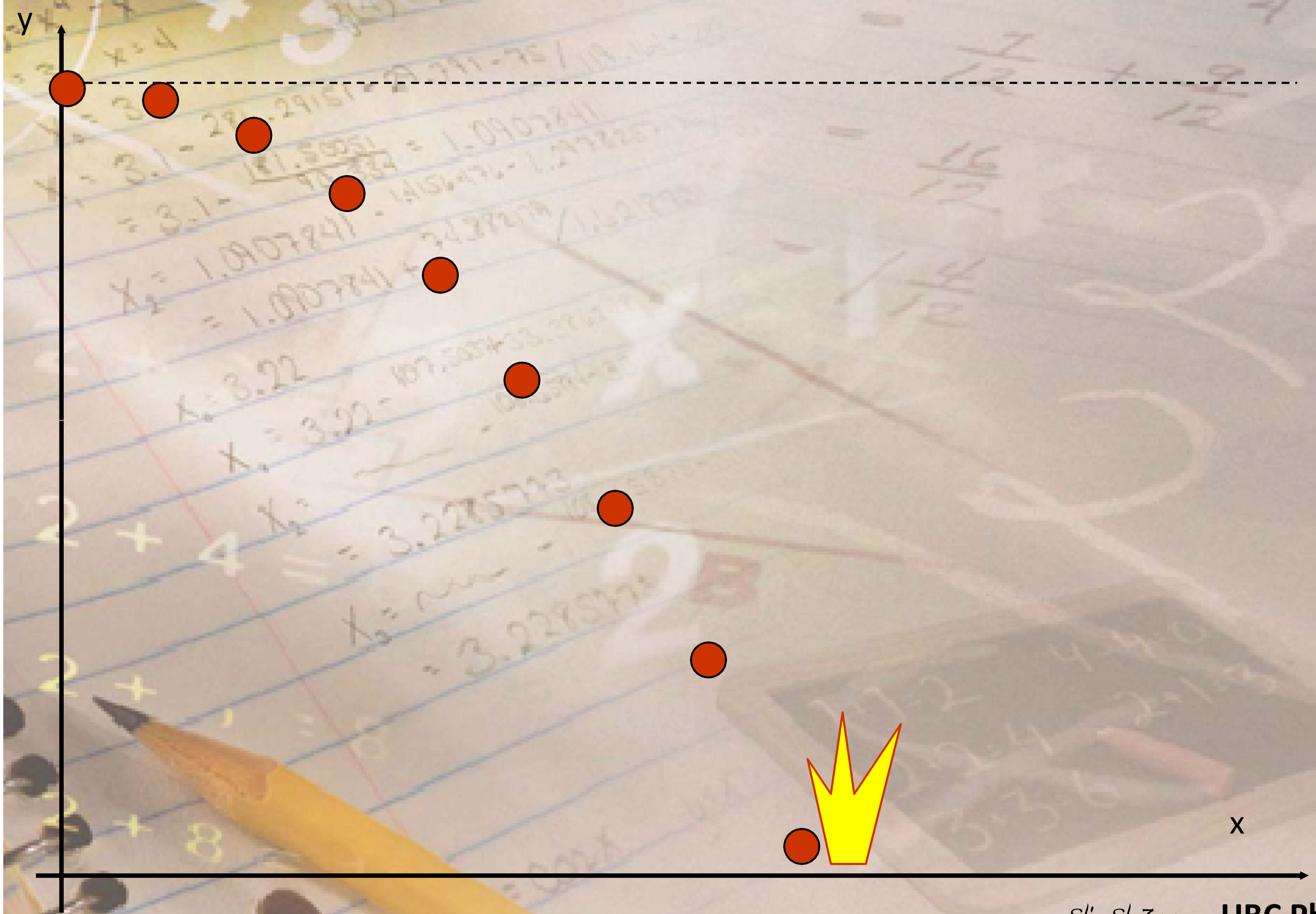
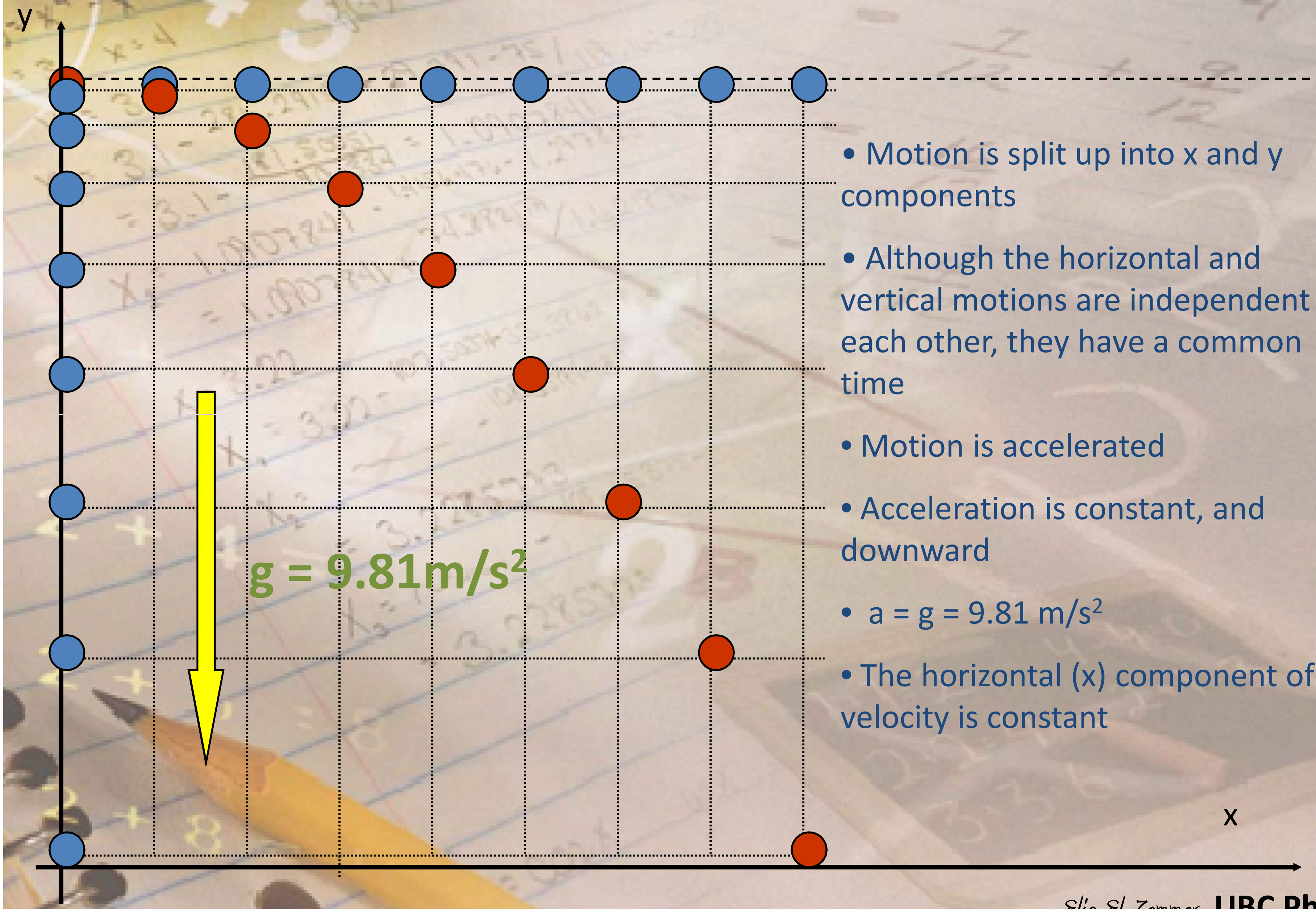
Motion of Objects Projected Horizontally



Motion of Objects Projected Horizontally



- Motion is split up into x and y components
- Although the horizontal and vertical motions are independent of each other, they have a common time
- Motion is accelerated
- Acceleration is constant, and downward
- $a = g = 9.81\ m/s^2$
- The horizontal (x) component of velocity is constant

Motion of Objects Projected Horizontally

ANALYSIS OF MOTION

ASSUMPTIONS:

- x-direction (horizontal): uniform motion
- y-direction (vertical): accelerated motion
- no air resistance

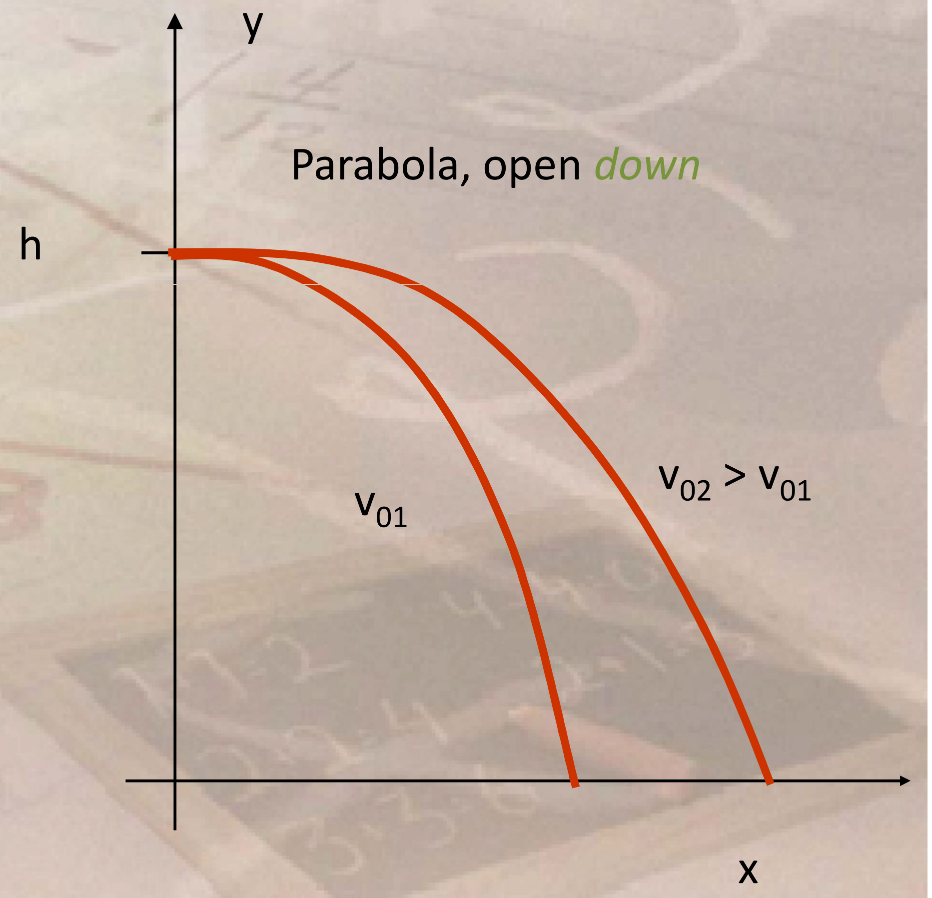
QUESTIONS:

- What is the trajectory?
- What is the total time of the motion?
- What is the horizontal range?
- What is the final velocity?

Motion of Objects Projected Horizontally

Trajectory

	X uniform motion	Y accelerated motion
Acceleration	$a_x = 0$	$a_y = g = 9.81$ m/s^2
Velocity	$v_x = v_0$	$v_y = g\,t$
Displacement	$x = v_0\,t$	$y = h + ½\,g\,t^2$



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Motion of Objects Projected Horizontally

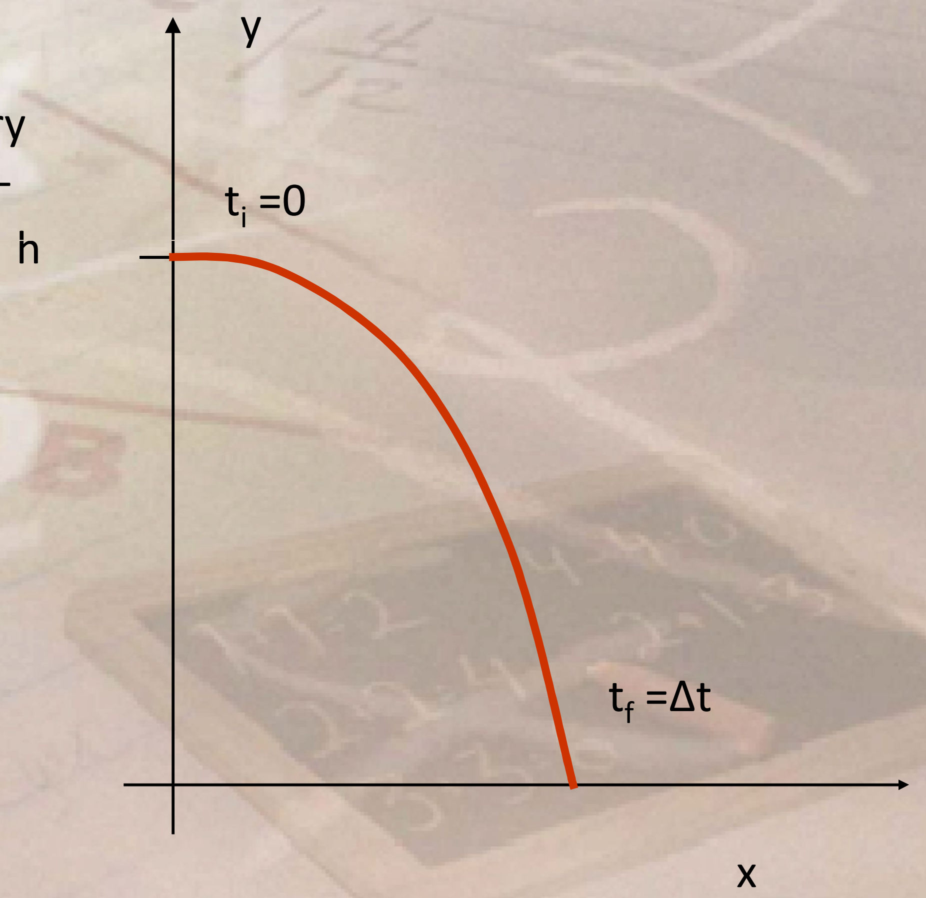
Total Time, Δt

The total time is simply the hang time of the projectile – that is to say the time it remains air born. So we only need to worry about the y-component (accelerated, non-constant).

$$d = v_0 t + \frac{1}{2} a t^2$$

Acceleration = 9.8 m/s²
Initial velocity = 0 m/s
Distance = height h

$$\sqrt{\frac{2h}{9.8}} = \sqrt{\frac{h}{4.9}} = t$$



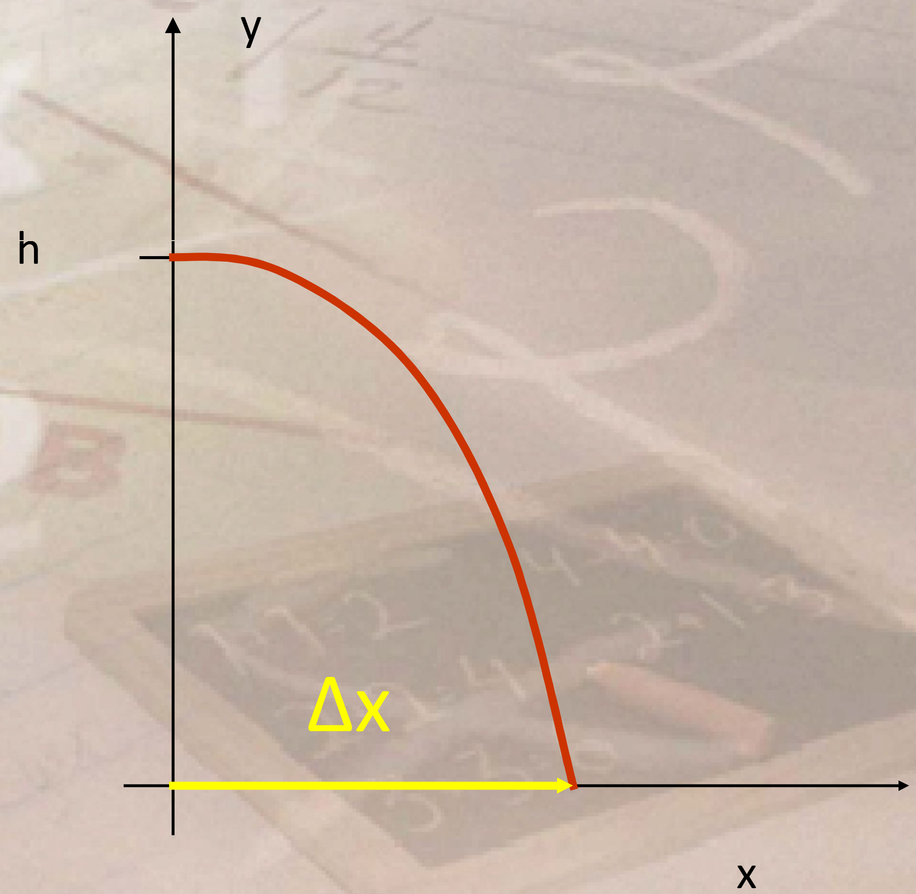
Motion of Objects Projected Horizontally

Horizontal Range, Δx

$$\Delta x = v_0\,\Delta t$$

Recall $\Delta t = \sqrt{\dfrac{h}{4.9}}$

$$\Delta x = v_o\sqrt{\frac{h}{4.9}}$$



Motion of Objects Projected Horizontally

Final Velocity

The final velocity in the x direction is constant → $v_x = v_0$

The final velocity in the y direction depends on the acceleration.

$$v_f = v_0 + at$$

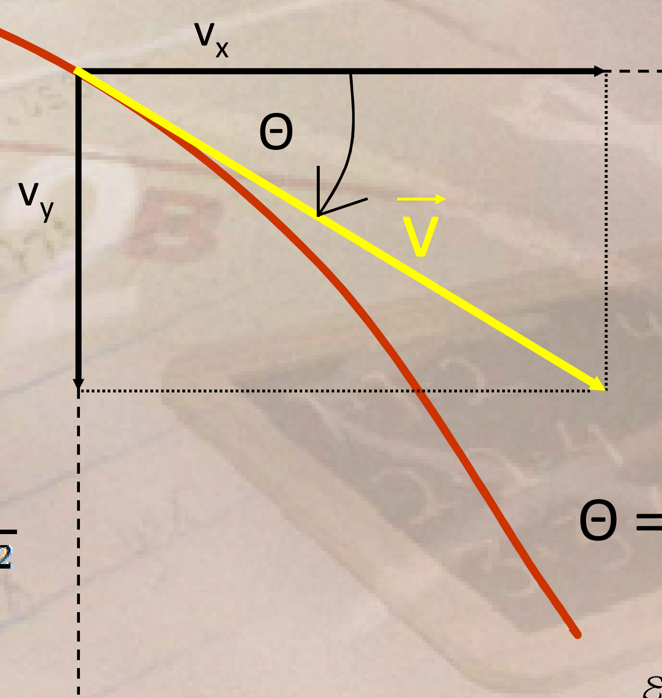
Initial velocity is zero.

$$v_y = 9.8t$$

Using Pythagorean theorem,

$$v_f = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + (9.8t)^2}$$

Where t is the hang time.



Θ is the angle between the projectile and the ground.

$$\Theta = \tan^{-1}(v_y/v_x)$$

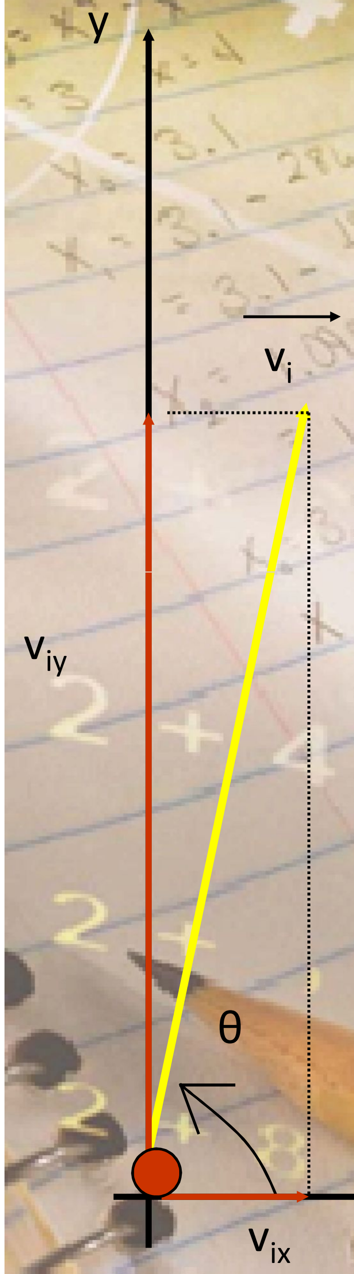
Motion of Objects Projected Horizontally

SUMMARY

h – initial height, v_0 – initial horizontal velocity, g – acceleration due to gravity $g= -9.81 m/s^2$

Trajectory	Half -parabola, open down
Total time	$\Delta t = \sqrt{\frac{h}{4.9}}$
Horizontal Range	$\Delta x = v_o \sqrt{\frac{h}{4.9}}$
Final Velocity	$v_f = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + (9.8t)^2}$ $\Theta = \tan^{-1} (v_y/v_x)$

Motion of Objects Projected at an Angle



Initial position: x = 0, y = 0

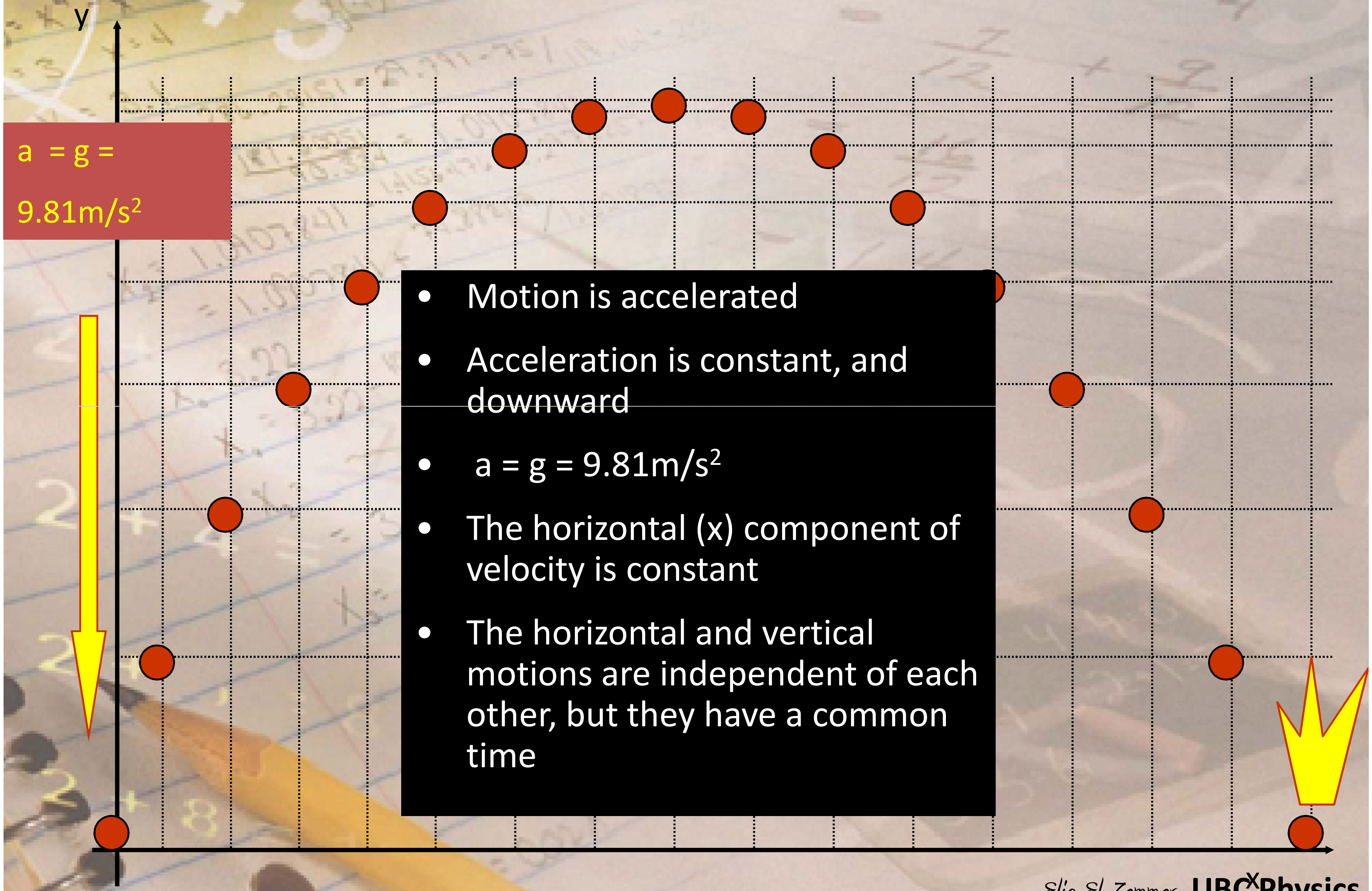
initial velocity: $\vec{vi} = v_i\ [\Theta]$

Velocity components:

x- direction : $v_{ix} = v_i \cos \Theta$

y- direction : $v_{iy} = v_i \sin \Theta$

Motion of Objects Projected at an Angle



- Motion is accelerated
- Acceleration is constant, and downward
- $a = g = 9.81 m/s^2$
- The horizontal (x) component of velocity is constant
- The horizontal and vertical motions are independent of each other, but they have a common time

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Motion of Objects Projected at an Angle

ANALYSIS OF MOTION:

ASSUMPTIONS

- x-direction (horizontal): uniform motion
- y-direction (vertical): accelerated motion
- no air resistance

QUESTIONS

- What is the trajectory?
- What is the total time of the motion?
- What is the horizontal range?
- What is the maximum height?
- What is the final velocity?

Motion of Objects Projected at an Angle

Equations of motion:

	X **Uniform motion**	**Y** **Accelerated motion**
ACCELERATION	$a_x = 0$	$a_y \ = g = 9.81\ m/s^2$
VELOCITY	$v_x = v_{ix} = v_i \cos \Theta$ $v_x = v_i \cos \Theta$	$v_y = v_{iy} + g\ t$ $v_y = v_i \sin \Theta + g\ t$
DISPLACEMENT	$x = v_{ix}\ t = v_i\ t \cos \Theta$ $x = v_i\ t \cos \Theta$	$y = h + v_{iy}\ t + ½\ g\ t^2$ $y = v_i\ t \sin \Theta + ½\ g\ t^2$

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Motion of Objects Projected at an Angle

Equations of motion:

	X **Uniform motion**	**Y** **Accelerated motion**
ACCELERATION	$a_x = 0$	$a_y = g = 9.81\ m/s^2$
VELOCITY	$v_x = v_i \cos \Theta$	$v_y = v_i \sin \Theta + g\,t$
DISPLACEMENT	$x = v_i\, t \cos \Theta$	$y = v_i\, t \sin \Theta + ½\, g\, t^2$

Motion of Objects Projected at an Angle

Trajectory

$x = v_i \, t \cos \Theta$

$y = v_i \, t \sin \Theta + \frac{1}{2} g t^2$

Eliminate time, t

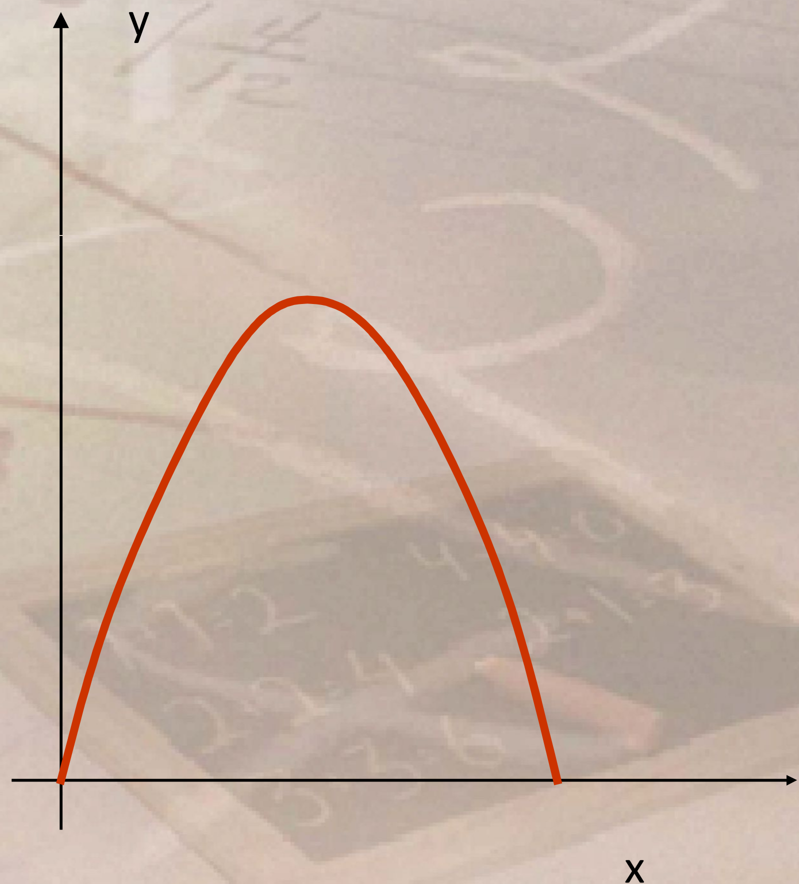
$t = x/(v_i \cos \Theta)$

$$y = \frac{v_i x \sin \Theta}{v_i \cos \Theta} + \frac{g x^2}{2 v_i^2 \cos^2 \Theta}$$

$$y = x \tan \Theta + \frac{g}{2 v_i^2 \cos^2 \Theta} x^2$$

$$\boxed{y = bx + ax^2}$$

Parabola, open *down*



Motion of Objects Projected at an Angle

Total Time, Δt

$v_f = v_i + at$

Initial velocity is in the y-direction. Final velocity is 0.

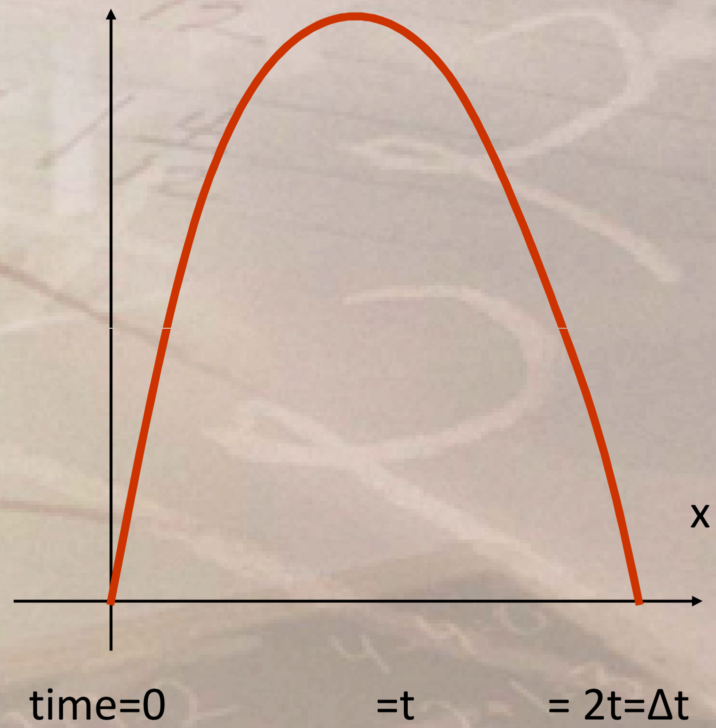
$0 = v_i t \sin \Theta - gt$

Solve for t:

$t = (v_i \sin \Theta)/g$

But the projectile doesn't just go up, it also comes down. So Δt = 2t

$$\boxed{\Delta t = \frac{2\, v_i \sin \Theta}{g}}$$

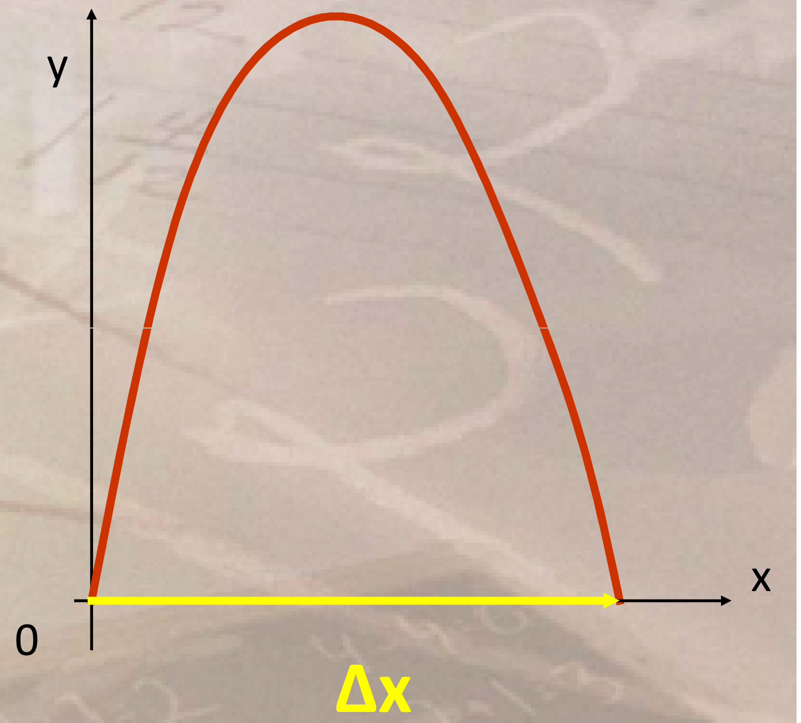


Motion of Objects Projected at an Angle

Horizontal Range, Δx

$x = v_i\, t \cos \Theta$

time is the total hang time Δt



$\Delta x = v_i\, \Delta t \cos \Theta$

$$\Delta t = \frac{2\, v_i \sin \Theta}{g}$$

$\sin(2\,\Theta) = 2 \sin \Theta \cos \Theta$

$$\Delta x = \frac{2 v_i^{\,2} \sin \Theta \cos \Theta}{g}$$

$$\boxed{\Delta x = \frac{v_i^{\,2} \sin(2\,\Theta)}{g}}$$

Motion of Objects Projected at an Angle

Horizontal Range, Δx

$$\Delta x = \frac{v_i^{\,2} \sin(2\,\Theta)}{g}$$

Θ (deg)	sin (2 Θ)
0	0.00
15	0.50
30	0.87
45	1.00
60	0.87
75	0.50
90	0

•CONCLUSIONS:

- Horizontal range is greatest for the throw angle of 45^0
- Horizontal ranges are the same for angles Θ and (90^0 – Θ)

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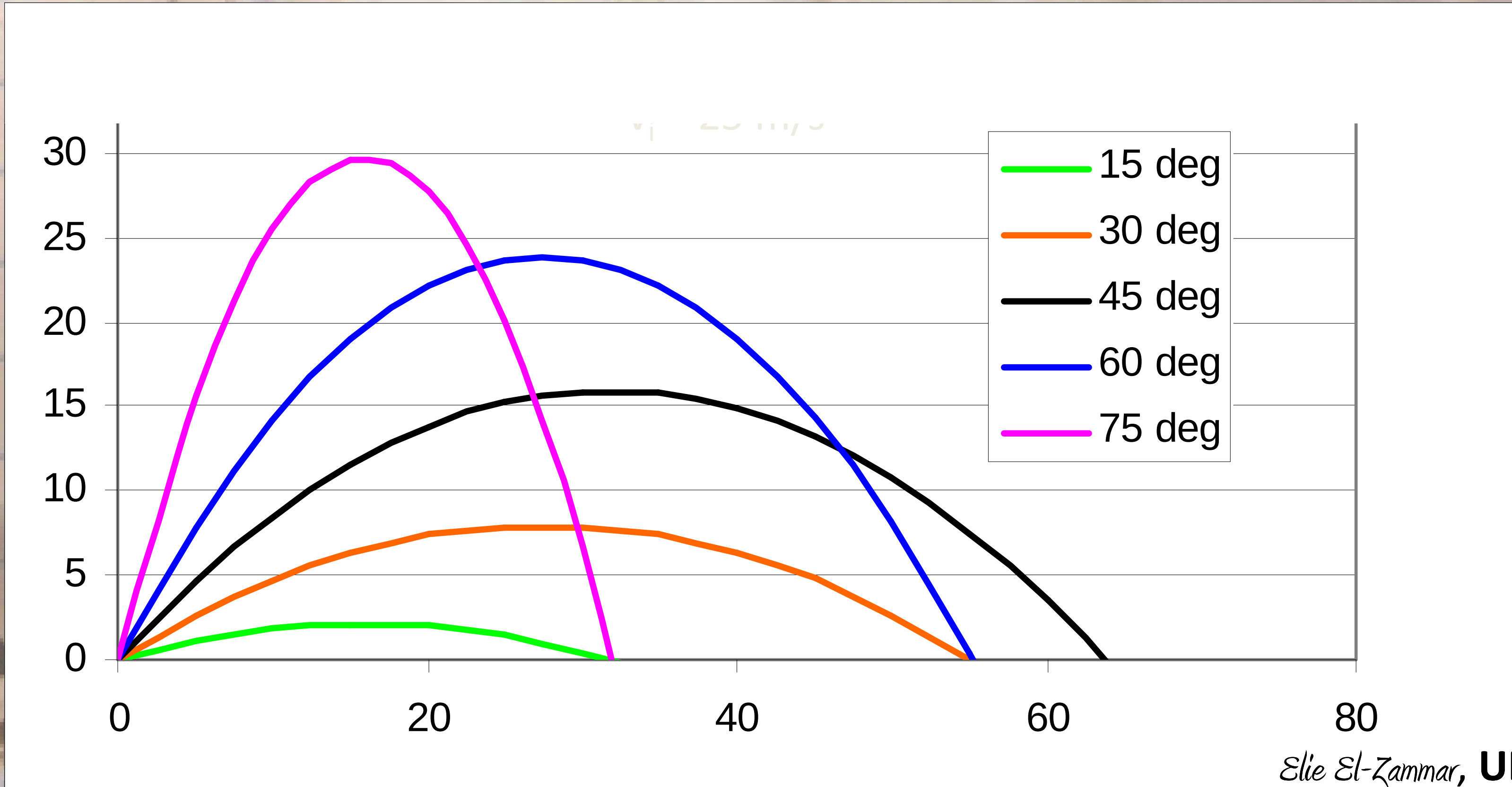
Motion of Objects Projected at an Angle

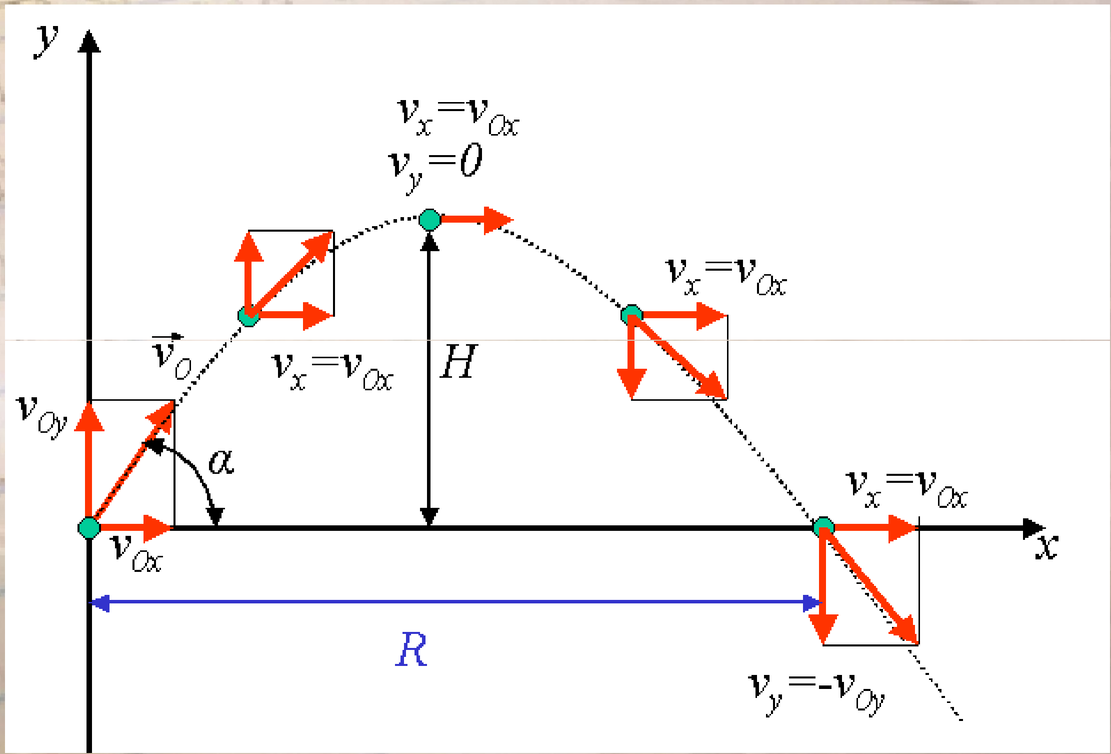
Trajectory and horizontal range

$$y = x\tan\Theta + \frac{g}{2v_i^2\cos^2\Theta}x^2$$



Motion of Objects Projected at an Angle

Velocity



- Final speed = initial speed *(conservation of energy)*
- Impact angle = - launch angle *(symmetry of parabola)*

Motion of Objects Projected at an Angle

Maximum Height

$v_y = v_i \sin \Theta - g\,t$

$d = v_i\, t \sin \Theta + ½\, a\, t^2$

At maximum height $v_y = 0$

$0 = v_i \sin \Theta + g\, t_{up}$

$$t_{up} = \frac{v_i \sin \Theta}{g}$$

$t_{up} = \Delta t/2$

$h_{max} = v_i\, t_{up} \sin \Theta - ½\, g\, t_{up}^{2}$

$h_{max} = v_i^2 \sin^2 \Theta/g - ½\, g(v_i^2 \sin^2 \Theta)/g^2$

$$\boxed{h_{max} = \frac{v_i^2 \sin^2 \Theta}{2g}}$$

Motion of Objects Projected at an Angle

Projectile Motion – Final Equations

Trajectory	Parabola, open down
Total time	$\Delta t = \dfrac{2\, v_i \sin \Theta}{g}$
Horizontal range	$\Delta x = \dfrac{v_i^{\,2} \sin (2\,\Theta)}{g}$
Max height	$h_{max} = \dfrac{v_i^2 \sin^2 \Theta}{2g}$

Motion of Objects Projected at an Angle

PROJECTILE MOTION - SUMMARY

- Projectile motion is motion with a constant horizontal velocity combined with a constant vertical acceleration
- The projectile moves along a parabola

Quick Review

- Since distance, velocity, range all depend on the acceleration of gravity, knowing the hang time is key in solving projectile motion problems.

- In order to visualize and compute projectile motion problems, we must separate the motion into x and y components.

- Projectiles launched at different angles will travel a different distance because of their respective hang times.

Demonstration

Now that we know how to predict the motion of a projectile, lets see how the projectile actually acts.

We will fire a projectile at an initial velocity v_i and we will solve for the distance traveled.

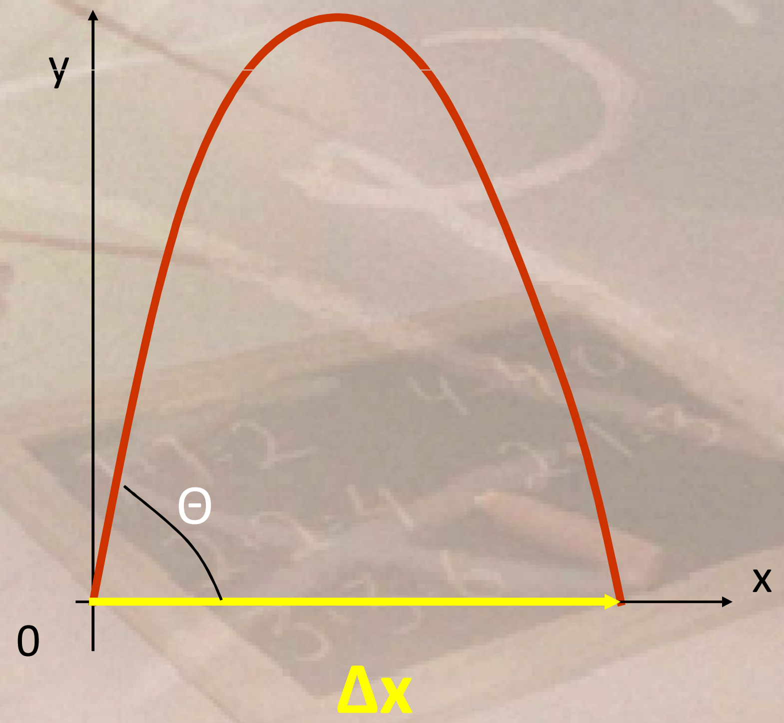
We will fire them at 3 different angles:

30^o

$45\ ^o$

$60\ ^o$

- Which will spend the most time in the air?
- Which will go the furthers?



Demonstration

What We Know

Initial Velocity = 8 m/s
Acceleration = 9.8 m/s²
Angle: 30º

What is the horizontal distance traveled?

5.66 m

But what about significant figures? Let us adjust our answer.

5.5 m +/- 0.5 m

Demonstration

What We Know

Initial Velocity = 8 m/s

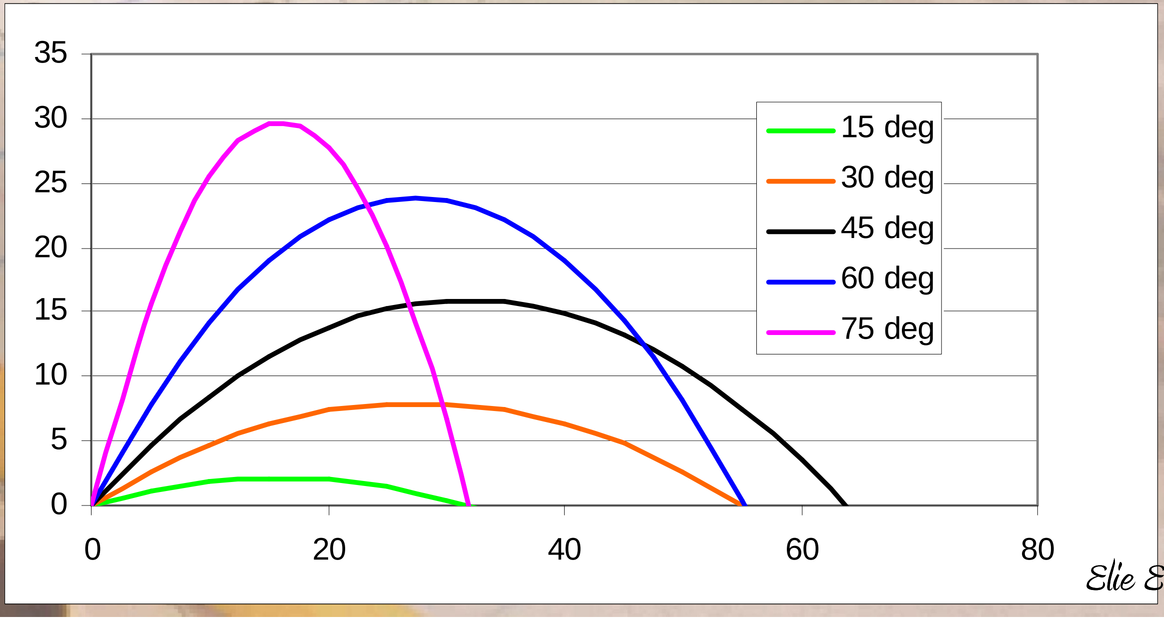
Acceleration = 9.8 m/s²

Angle: 60°

What is the horizontal distance traveled?

5.5 m +/- 0.5 m

Why is this the same distance as the 60° angle?



Demonstration

What We Know

Initial Velocity = 8 m/s

Acceleration = 9.8 m/s^2

Angle: 45^o

What is the hang time?

1.15 seconds

What is the horizontal distance traveled?

6.5 m +/- 0.5 m

What is the vertical distance traveled?

1.63 m

Demonstration

LETS TRY IT

We will fire 3 projectiles and average the results. We will do this the 3 angles previously discussed.

Error Analysis

What kind of error could of have altered out results?

- Measurement (angle, psi, tape measure)
- Air resistance
- Apparatus: In fact, the projectile actually doesn’t fire at 8 m/s . It fires at a much faster velocity but it is slowed down by friction.

What is the biggest problem with this system?

Consistency! Because of all the factors that affect the projectile, it is very difficult to get consistent results. We must place the ball the same distance into the tube, we must assure the proper amount of air pressure, there must be no leaks, the ball must be perfectly smooth and be the same roughness for EVERY launch to insure accurate and consistent results. For this reason we must correct for the error by taking multiple measurements and averaging the results.

Real Life Equations

- When you further study projectile motion and kinematics you will be able to actually predict the motion of projectiles. It is possible to consider the impact of air resistance and even the spin of the earth. What do these 3 dimensional equations looks like?

Real Velocity

So if the projectile wasn’t traveling at exactly 8 m/s when it came out of the cannon, how fast was it traveling?

Real Life Applications

- Bullets on a Straight Spinning Flight
- Curve Balls, Dimpled Golf Balls, and Other Tricks With Spin
- Powered Projectiles: Rockets and Missiles